PROGRAM TEKNOLOGI INFORMASI DAN ILMU KOMPUTER

TRANSFORMASI-Z

- Transformsi-Z Langsung
- Sifat-sifat Transformasi-Z
- Transformasi -Z Rasional
- Transformasi-Z Balik
- Transformasi-Z Satu Sisi

TRANSFORMASI-Z LANGSUNG

- **Definisi :**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Contoh Soal 1

Tentukan transformasi Z dari beberapa sinyal diskrit di bawah ini

$$a).\ x_1(n) = \{1,\ 2,\ 5,\ 7,\ 0,\ 1\}$$

$$b).\ x_2(n) = \{1,\ 2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$$

$$c).\ x_3(n) = \{0,\ 1,\ 2,\ 5,\ 7,\ 0,\ 1\}$$

$$d).\ x_4(n) = \{2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$$

Jawab:

$$a).\ x_1(n) = \{1,\ 2,\ 5,\ 7,\ 0,\ 1\}$$

$$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$b).\ x_2(n) = \{1,\ 2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$$

$$X_2(z) = z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$$

$$c).\ x_3(n) = \{0, 1, 2, 5, 7, 0, 1\}$$

$$X_3(z) = z^{-1} + 2z^{-2} + 5z^{-3} + 7z^{-4} + z^{-6}$$

$$d).\ x_4(n) = \{2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$$

$$X_4(z) = 2z^1 + 5 + 7z^{-1} + z^{-3}$$

Contoh Soal 2

Tentukan transformasi Z dari beberapa sinyal impuls di bawah ini

$$a).\ x_1(n) = \delta(n)$$

$$b).\ x_2(n) = \delta(n-k),\ k > 0$$

$$c).\ x_3(n) = \delta(n+k),\ k > 0$$

Jawab:

$$a).\ X_1(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = 1$$

$$b).\ X_2(z) = \sum_{n=-\infty}^{\infty} \delta(n-k) z^{-n} = z^{-k}$$

$$c).\ X_3(z) = \sum_{n=-\infty}^{\infty} \delta(n+k) z^{-n} = z^{k}$$

Contoh Soal 3

Tentukan transformasi Z dari sinyal $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Jawab:

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \sum_{n=0}^{\infty} (A)^n$$

$$= 1 + A + A^2 + A^3 + \cdots = \frac{1}{1 - A} \qquad |A| < 1$$

$$\left|\frac{1}{2} z^{-1}\right| < 1 \quad \rightarrow \quad |z| > \frac{1}{2} \quad \rightarrow \quad X(z) = \frac{1}{1 - \left(\frac{1}{2}\right) z^{-1}}$$

$$x(n) = \alpha^n u(n) \quad \rightarrow \quad X(z) = \frac{1}{1 - \alpha z^{-1}}$$

$$x(n) = u(n) \quad \rightarrow \quad X(z) = \frac{1}{1 - z^{-1}}$$

SIFAT-SIFAT TRANSFORMASI-Z

- **Linieritas**

$$x_1(n) \to X_1(z) \qquad\qquad x_2(n) \to X_2(z)$$

$$x(n) = a_1x_1(n) + a_2x_2(n) \quad \to \quad X(z) = a_1X_1(z) + a_2X_2(z)$$

Contoh Soal 4

Tentukan transformasi Z dari sinyal $x(n) = \left[3(2)^n - 4(3)^n\right]u(n)$

Jawab:

$$x_1(n) = (2)^n u(n) \qquad\qquad x_2(n) = (3)^n u(n)$$

$$X(z) = 3X_1(z) - 4X_2(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

Contoh Soal 5

Tentukan transformasi Z dari sinyal-sinyal di bawah ini :

a). $x(n) = \cos(\omega_o n)\, u(n)$

b). $x(n) = \sin(\omega_o n)\, u(n)$

Jawab:

$$a).\ x(n) = \cos(\omega_o n)\, u(n) = \frac{1}{2} e^{j\omega_o n} u(n) + \frac{1}{2} e^{-j\omega_o n} u(n)$$

$$X(z) = \frac{1}{2}\frac{1}{1 - e^{j\omega_o} z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega_o} z^{-1}}$$

$$X(z) = \frac{1}{2}\frac{(1 - e^{-j\omega_o}) + (1 - e^{+j\omega_o})}{(1 - e^{j\omega_o} z^{-1})(1 - e^{-j\omega_o} z^{-1})} = \frac{1}{2}\frac{(1 - e^{+j\omega_o} z^{-1} - e^{-j\omega_o} z^{-1} + 1)}{(1 - e^{+j\omega_o} z^{-1} - e^{-j\omega_o} + z^{-2})}$$

$$x(n) = \cos\omega_o n \quad \rightarrow \quad X(z) = \frac{1 - z^{-1}\cos\omega_o}{1 - 2z^{-1}\cos\omega_o + z^{-2}}$$

b). $x(n) = \sin(\omega_o n)\,u(n) = \dfrac{1}{2j} e^{j\omega_o n} u(n) - \dfrac{1}{2j} e^{-j\omega_o n} u(n)$

$$X(z) = \frac{1}{2j}\frac{1}{1 - e^{j\omega_o} z^{-1}} - \frac{1}{2j}\frac{1}{1 - e^{-j\omega_o} z^{-1}}$$

$$X(z) = \frac{1}{2j}\frac{(1 - e^{-j\omega_o}) - (1 - e^{+j\omega_o})}{(1 - e^{j\omega_o} z^{-1})(1 - e^{-j\omega_o} z^{-1})}$$

$$= \frac{1}{2j}\frac{(e^{+j\omega_o} z^{-1} - e^{-j\omega_o} z^{-1})}{(1 - e^{+j\omega_o} z^{-1} - e^{-j\omega_o} + z^{-2})}$$

$$x(n) = \sin\omega_o n \quad \rightarrow \quad X(z) = \frac{z^{-1}\sin\omega_o}{1 - 2z^{-1}\cos\omega_o + z^{-2}}$$

Scaling in the Z-domain

$$x(n) \rightarrow X(z) \qquad a^n x_1(n) \rightarrow X(a^{-1}z) = X\left(\frac{z}{a}\right)$$

Contoh Soal 6

Tentukan transformasi Z dari sinyal-sinyal di bawah ini :

$$a).\ x_1(n) = a^n \cos(\mu_o n) \qquad b).\ x_2(n) = a^n \sin(\mu_o n)$$

Jawab:

$$x(n) = \cos\omega_o n \rightarrow X(z) = \frac{1 - z^{-1}\cos\omega_o}{1 - 2z^{-1}\cos\omega_o + z^{-2}}$$

$$x_1(n) = a^n \cos\omega_o n \rightarrow X_1(z) = \frac{1 - (a^{-1}z)^{-1}\cos\omega_o}{1 - 2(a^{-1}z)^{-1}\cos\omega_o + (a^{-1}z)^{-2}}$$

$$X_1(z) = \frac{1 - az^{-1}\cos\omega_o}{1 - 2az^{-1}\cos\omega_o + a^2 z^{-2}} \qquad X_2(z) = \frac{az^{-1}\sin\omega_o}{1 - 2az^{-1}\cos\omega_o + a^2 z^{-2}}$$

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Time Reversal

$$x(n) \rightarrow X(z) \qquad x(-n) \rightarrow X(z^{-1})$$

Contoh Soal 7

Tentukan transformasi Z dari sinyal $x(n) = u(-n)$

Jawab:

$$x(n) = u(n) \quad \rightarrow \quad X(z) = \frac{1}{1 - z^{-1}}$$

$$x(n) = u(-n) \quad \rightarrow \quad X(z) = \frac{1}{1 - (z^{-1})^{-1}} = \frac{1}{1 - z}$$

$$x(n) = u(-n) \quad \rightarrow \quad X(z) = \frac{1}{1 - z}$$

Diferensiasi dalam domain z

$$x(n) \rightarrow X(z) \qquad nx(n) \rightarrow -z\frac{dX(z)}{dz}$$

Contoh Soal 8

Tentukan transformasi Z dari sinyal $x(n) = na^n u(n)$

Jawab:

$$x_1(n) = a^n u(n) \rightarrow X_1(z) = \frac{1}{1-az^{-1}}$$

$$x(n) = na^n u(n) \rightarrow X(z) = -z\frac{dX_1(z)}{dz}$$

$$X(z) = -z\frac{dX_1(z)}{dz} = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = (-z)\frac{-az^{-2}}{\left(1-az^{-1}\right)^2}$$

$$na^n u(n) \rightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \qquad nu(n) \rightarrow \frac{z^{-1}}{\left(1-z^{-1}\right)^2}$$

- # Konvolusi antara dua sinyal

$$x_1(n) \rightarrow X_1(z) \qquad x_2(n) \rightarrow X_2(z)$$

$$x(n) = x_1(n) * x_2(n) \rightarrow X(z) = X_1(z)X_2(z)$$

Contoh Soal 9

Tentukan konvolusi antara $x_1(n)$ dan $x_2(n)$ dengan :

$$x_1(n) = \{1, -2, 1\} \qquad x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{lainnya} \end{cases}$$

Jawab:

$$X_1(z) = 1 - 2z^{-1} + z^{-2} \qquad X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)X_2(z) = (1 - 2z^{-1} + z^{-2})(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$x(n) = x_1(n) * x_2(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$$

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TRANSFORMASI Z RASIONAL

- **Pole dan Zero**

Pole : harga-harga $z = p_i$ yang menyebabkan $X(z) = \infty$

Zero : harga-harga $z = z_i$ yang menyebabkan $X(z) = 0$

- **Fungsi Rasional**

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_o + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_o + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$a_o \neq 0 \;\; b_o \neq 0 \quad \rightarrow \quad X(z) = \frac{N(z)}{D(z)} = \frac{b_o z^{-M}}{a_o z^{-N}} \, \frac{z^M + \left(\frac{b_1}{b_o}\right) z^{-M-1} + \cdots + \left(\frac{b_M}{b_o}\right)}{Z^N + \left(\frac{a_1}{a_o}\right) z^{N-1} + \cdots + \left(\frac{a_N}{a_o}\right)}$$

$$a_o \neq 0 \;\; b_o \neq 0 \quad \rightarrow \quad X(z) = \frac{N(z)}{D(z)} = \frac{b_o z^{-M}}{a_o z^{-N}} \; \frac{z^{M} + \left(\frac{b_1}{b_o}\right) z^{-M-1} + \cdots + \left(\frac{b_M}{b_o}\right)}{Z^{N} + \left(\frac{a_1}{a_o}\right) z^{N-1} + \cdots + \left(\frac{a_N}{a_o}\right)}$$

- **N(z) dan D(z) polinom**

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_o}{a_o} z^{N-M} \; \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

$$X(z) = G\, z^{N-M} \; \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

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Contoh Soal 10

Tentukan pole dan zero dari $X(z) = \dfrac{2 - 1{,}5z^{-1}}{1 - 1{,}5z^{-1} + 0{,}5z^{-2}}$

Jawab:

$$X(z) = \frac{2}{1}\,\frac{z^{-1}}{z^{-2}}\,\frac{z - 0{,}75}{z^{2} - 1{,}5z + 0{,}5}$$

$$= 2z^{2-1}\,\frac{z - 0{,}75}{(z-1)(z-0{,}5)} = \frac{2z(z-0{,}75)}{(z-1)(z-0{,}5)}$$

Zero : $z_1 = 0 \qquad z_2 = 0{,}75$

Pole : $p_1 = 1 \qquad p_2 = 0{,}5$

Contoh Soal 11

Tentukan pole dan zero dari $$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0{,}5z^{-2}}$$

Jawab:

$$X(z) = \frac{z(z+1)}{z^2 - z + 0{,}5}$$

$$= \frac{z(z+1)}{[z - (0{,}5 + j0{,}5)][z - (0{,}5 - j0{,}5)]}$$

Zero : $z_1 = 0 \qquad z_2 = 1$

Pole : $p_1 = 0{,}5 + j0{,}5 \qquad p_2 = 0{,}5 - j0{,}5 \quad \Longrightarrow \quad p_1 = p_2^*$

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos \omega_0 n)u(n)$	$\frac{1-z^{-1}\cos \omega_0}{1-2z^{-1}\cos \omega_0 + z^{-2}}$	$\|z\| > 1$
8	$(\sin \omega_0 n)u(n)$	$\frac{z^{-1}\sin \omega_0}{1-2z^{-1}\cos \omega_0 + z^{-2}}$	$\|z\| > 1$
9	$(a^n \cos \omega_0 n)u(n)$	$\frac{1-az^{-1}\cos \omega_0}{1-2az^{-1}\cos \omega_0 + a^2 z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n \sin \omega_0 n)u(n)$	$\frac{az^{-1}\sin \omega_0}{1-2az^{-1}\cos \omega_0 + a^2 z^{-2}}$	$\|z\| > \|a\|$

Fungsi Sistem dari Sistem LTI

$$y(n) = h(n) * x(n) \;\rightarrow\; Y(z) = H(z)X(z) \;\rightarrow\; H(z) = \frac{Y(z)}{X(z)}$$

Respon impuls $h(n) \rightarrow H(z)$ **Fungsi sistem**

Persamaan beda dari sistem LTI :

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z)[1 + \sum_{k=1}^{N} a_k z^{-k}] = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = H(z)$$

Fungsi sistem rasional

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = H(z)$$

pole-zero system

Hal khusus I : a_k = 0, 1 á k á N

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

All-zero system

Hal khusus II : b_k = 0, 1 á k á M

$$H(z) = \frac{b_o}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_o}{\sum_{k=0}^{N} a_k z^{-k}} \qquad a_o = 1$$

All-pole system

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Contoh Soal 12

Tentukan fungsi sistem dan respon impuls sistem LTI :

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

Jawab:

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$Y(z)(1 - \frac{1}{2} z^{-1}) = 2X(z)$$

$$H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}} \quad \Longrightarrow \quad h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

TRANSFORMASI -Z BALIK

- **Definisi transformasi balik**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Teorema residu Cauchy :

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_o)^k} dz = \begin{cases} \frac{1}{(k-1)!} \left. \frac{d^{k-1} f(z)}{dz^{k-1}} \right|_{z=z_o}, & \text{bila } z_o \text{ di dalam } C \\ 0, & \text{bila } z_o \text{ di luar } C \end{cases}$$

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Ekspansi deret dalam z dan z^{-1}

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

Contoh Soal 13

Tentukan transformasi-z balik dari

$$X(z) = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4}\cdots$$

Jawab:

$$X(z) = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4}\cdots$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow x(n) = \left\{ \underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \cdots \right\}$$

- **Ekspansi fraksi-parsial dan tabel transformasi-z**

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_K X_K(z)$$

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_K x_K(n)$$

Contoh Soal 14

Tentukan transformasi-z balik dari $X(z) = \dfrac{1}{1 - 1{,}5z^{-1} + 0{,}5z^{-2}}$

Jawab:

$$X(z) = \frac{z^2}{z^2 - 1{,}5z + 0{,}5} = \frac{z^2}{(z-1)(z-0{,}5)}$$

$$\frac{X(z)}{z} = \frac{z}{z^2 - 1{,}5z + 0{,}5} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-0{,}5)}$$

$$\frac{X(z)}{z} = \frac{z}{z^2 - 1{,}5z + 0{,}5} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-0{,}5)} = \frac{2}{(z-1)} - \frac{1}{(z-0{,}5)}$$

$$\frac{z}{z^2 - 1{,}5z + 0{,}5} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-0{,}5)} = \frac{A_1(z-0{,}5) + A_2(z-1)}{(z-1)(z-0{,}5)}$$

$$\frac{z}{z^2 - 1{,}5z + 0{,}5} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-0{,}5)} = \frac{(A_1 + A_2)z - (0{,}5A_1 + A_2)}{z^2 - 1{,}5z + 0{,}5}$$

$$A_1 + A_2 = 1 \qquad 0{,}5A_1 + A_2 = 0 \quad \rightarrow \quad A_2 = -0{,}5A_1$$

$$A_1 - 0{,}5A_1 = 0{,}5A_1 = 1 \quad \rightarrow \quad A_1 = 2 \quad \rightarrow \quad A_2 = -1$$

$$X(z) = \frac{2}{(1 - z^{-1})} - \frac{1}{(1 - 0{,}5z^{-1})} \quad \Longrightarrow \quad x(n) = [2 - (0{,}5)^2]u(n)$$

Contoh Soal 15

Tentukan respon impuls dari suatu sistem LTI (Linear Time Invariant) yang dinyatakan oleh persamaan beda :

$$y(n) + 3y(n-1) + y(n-2) = 4{,}5x(n) + 9{,}5x(n-1)$$

Jawab:

$$Y(z) + 3z^{-1}Y(z) + 2z^{-2}Y(z) = 4{,}5X(z) + 9{,}5z^{-1}X(z)$$

$$Y(z)(1 + 3z^{-1} + 2z^{-2}) = X(z)(4{,}5 + 9{,}5z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4{,}5 + 9{,}5z^{-1}}{1 + 3z^{-1} + 2z^{-2}}$$

$$H(z) = \frac{4{,}5 + 9{,}5z^{-1}}{1 + 3z^{-1} + 2z^{-2}} \quad \longrightarrow \quad \frac{H(z)}{z} = \frac{4{,}5z + 9{,}5}{z^2 + 3z + 2}$$

$$\frac{H(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{z+2} = \frac{5}{z+1} - \frac{0{,}5}{z+2}$$

$$H(z) = \frac{5}{1-(-1)z^{-1}} - \frac{0{,}5}{1-(-2)z^{-1}}$$

$$h(n) = [5(-1)^n - 0{,}5(-2)^n]u(n)$$

Contoh Soal 16

Tentukan output dari suatu sistem LTI (Linear Time Invariant) yang dinyatakan oleh persamaan beda :

$$y(n) + 3y(n-1) + y(n-2) = 4{,}5x(n) + 9{,}5x(n-1)$$

$$y(-1) = 0 \qquad y(-2) = 0$$

dan mendapat input $x(n) = (-3)^n u(n)$ ⟹ $y(n) = y_{zs}(n)$

Jawab:

$$Y(z) + 3z^{-1}Y(z) + 2z^{-2}Y(z) = 4{,}5X(z) + 9{,}5z^{-1}X(z)$$

$$Y(z)(1 + 3z^{-1} + 2z^{-2}) = X(z)(4{,}5 + 9{,}5z^{-1})$$

$$Y(z)(1 + 3z^{-1} + 2z^{-2}) = (4,5 + 9,5z^{-1})X(z)$$

$$x(n) = (-3)^n u(n) \quad \rightarrow \quad X(z) = \frac{1}{1 - (-3)z^{-1}} = \frac{1}{1 + 3z^{-1}}$$

$$Y(z)(1 + 3z^{-1} + 2z^{-2}) = (4,5 + 9,5z^{-1}) \frac{1}{1 + 3z^{-1}}$$

$$Y(z) = \frac{(4,5 + 9,5z^{-1})}{(1 + 3z^{-1} + 2z^{-2})(1 + 3z^{-1})}$$

$$\frac{Y(z)}{z} = \frac{z^2(4,5 + 9,5z^{-1})}{z^3(1 + 3z^{-1} + 2z^{-2})(1 + 3z^{-1})}$$

$$\frac{Y(z)}{z} = \frac{(4,5z^2 + 9,5z)}{(z^2 + 3z + 2)(z + 3)} = \frac{(4,5z^2 + 9,5z)}{(z + 1)(z + 2)(z + 3)}$$

$$\frac{Y(z)}{z} = \frac{(4{,}5z^2 + 9{,}5z)}{(z^2 + 3z + 2)(z + 3)} = \frac{(4{,}5z^2 + 9{,}5z)}{(z + 1)(z + 2)(z + 3)}$$

$$\frac{(4{,}5z^2 + 9{,}5z)}{(z + 1)(z + 2)(z + 3)} = \frac{A_1}{(z + 1)} + \frac{A_2}{(z + 2)} + \frac{A_3}{(z + 3)}$$

$$\frac{Y(z)}{z} = \frac{A_1(z^2 + 5z + 6) + A_2(z^2 + 4z + 3) + A_3(z^2 + 3z + 2)}{(z + 1)(z + 2)(z + 3)}$$

$$A_1 + A_2 + A_3 = 4{,}5$$
$$5A_1 + 4A_2 + 3A_3 = 9{,}5$$
$$6A_1 + 3A_2 + 2A_3 = 0$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{vmatrix} = 2$$

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$$A_1 = \frac{\begin{vmatrix} 4,5 & 1 & 1 \\ 9,5 & 4 & 3 \\ 0 & 3 & 2 \end{vmatrix}}{D} = \frac{-5}{2} = -2,5 \qquad A_2 = \frac{\begin{vmatrix} 1 & 4,5 & 1 \\ 5 & 9,5 & 3 \\ 6 & 0 & 2 \end{vmatrix}}{D} = \frac{2}{2} = 1$$

$$-2,5 + 1 + A_3 = 4,5 \quad \rightarrow \quad A_3 = 6$$

$$\frac{Y(z)}{z} = \frac{-2,5}{(z+1)} + \frac{1}{(z+2)} + \frac{6}{(z+3)}$$

$$Y(z) = \frac{-2,5}{(1+z^{-1})} + \frac{1}{(1+2z^{-1})} + \frac{6}{(1+3z^{-1})}$$

$$y_{zs}(n) = [-2,5(-1)^n + (-2)^2 + 6(-3)^n]u(n)$$

- **Pole-pole berbeda semua**

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \cdots + \frac{A_k}{z - p_k} + \cdots + \frac{A_N}{z - p_N}$$

$$\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}$$

$$\left.\frac{(z - p_k)X(z)}{z}\right|_{z=p_k} = A_k$$

Contoh Soal 17

Tentukan zero-state response dari suatu sistem LTI yang mendapat input x(n) = u(n) dan dinyatakan oleh persamaan beda :

$$y(n) + 6y(n-1) + 8y(n-2) = 5x(n) - 28x(n-1) + 8x(n-2)$$

Jawab:

$$Y(z) + 6z^{-1}Y(z) + 8z^{-2}Y(z) = 5X(z) - 28z^{-1}X(z) + 8z^{-2}X(z)$$

$$X(z) = \frac{1}{1-z^{-1}} \quad \Longrightarrow \quad Y(z) = \frac{(5 - 28z^{-1} + 8z^{-2})}{1 + 6z^{-1} + 8z^{-2}} \frac{1}{1-z^{-1}}$$

$$\frac{Y(z)}{z} = \frac{(5z^2 - 28z + 8)}{(z^2 + 6z + 8)(z-1)} = \frac{A_1}{z+2} + \frac{A_2}{z+4} + \frac{A_3}{z-1}$$

$$\frac{Y(z)}{z} = \frac{(5z^2 - 28z + 8)}{(z+2)(z+4)(z-1)} = \frac{A_1}{z+2} + \frac{A_2}{z+4} + \frac{A_3}{z-1}$$

$$A_1 = \frac{(z+2)Y(z)}{z} = \left.\frac{5z^2 - 28z + 8}{(z+4)(z-1)}\right|_{z=-2} = \frac{20 + 56 + 8}{(2)(-3)} = \frac{84}{-6} = -14$$

$$A_2 = \frac{(z+4)Y(z)}{z} = \left.\frac{5z^2 - 28z + 8}{(z+2)(z-1)}\right|_{z=-4} = \frac{80 + 112 + 8}{(-2)(-5)} = \frac{200}{10} = 20$$

$$A_3 = \frac{(z-1)Y(z)}{z} = \left.\frac{5z^2 - 28z + 8}{(z+2)(z+4)}\right|_{z=1} = \frac{5 - 28 + 8}{(3)(5)} = \frac{-15}{15} = -1$$

$$\frac{Y(z)}{z} = \frac{-14}{z+2} + \frac{20}{z+4} + \frac{-1}{z-1} \qquad Y(z) = \frac{-14}{1+2z^{-1}} + \frac{20}{1+4z^{-1}} + \frac{-1}{1-z^{-1}}$$

$$y_{zs}(n) = [-14(-2)^n + 20(-4)^n - 1]u(n)$$

- **Ada dua pole yang semua**

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \cdots + \frac{A_{1k}}{(z - p_k)^2} + \frac{A_{2k}}{z - p_k} + \cdots + \frac{A_N}{z - p_N}$$

$$A_{1k} = \left. \frac{(z - p_k)^2 X(z)}{z} \right|_{z=p_k}$$

$$A_{2k} = \frac{d}{dz}\left[\frac{(z - p_k)^2 X(z)}{z} \right]_{z=p_k}$$

Contoh Soal 18

Tentukan transformasi-Z balik dari :

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Jawab:

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{(z-1)^2} + \frac{A_3}{(z-1)}$$

$$A_1 = \frac{(z+1)X(z)}{z} = \left.\frac{z^2}{(z-1)^2}\right|_{z=-1} = \frac{1}{4}$$

$$A_2 = \frac{(z-1)^2 X(z)}{z} = \left.\frac{z^2}{(z+1)}\right|_{z=1} = \frac{1}{2}$$

$$A_3 = \frac{d}{dz}\left[\frac{(z-1)^2 X(z)}{z}\right] = \frac{d}{dz}\left[\frac{z^2}{(z+1)}\right]$$

$$= \frac{(2z)(z+1) - (1)(z^2)}{(z+1)^2} = \left.\frac{z^2 + 2z}{(z+1)^2}\right|_{z=1} = \frac{3}{4}$$

$$x(n) = [\frac{1}{4}(-1)^n + \frac{1}{2}n + \frac{3}{4}]u(n)$$

- **Pole kompleks**

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

$$p_1 = p \quad \rightarrow \quad p_2 = p^* \qquad\qquad A_1 = A \quad \rightarrow \quad A_2 = A^*$$

$$\frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}} = \frac{A - Ap^* z^{-1} + A^* - A^* pz^{-1}}{1 - pz^{-1} - p^* z^{-1} + pp^* z^{-2}}$$

$$\frac{(A + A^*) - (Ap^* + A^* p)z^{-1}}{1 - (p + p^*)z^{-1} + pp^* z^{-2}} = \frac{b_o + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$A + A^* = \mathrm{Re}(A) + j\,\mathrm{Im}(A) + \mathrm{Re}(A) - j\,\mathrm{Im}(A) = 2\,\mathrm{Re}(A)$$

$$b_o = A + A^* = 2\,\mathrm{Re}(A)$$

$$p + p^* = \mathrm{Re}(p) + j\,\mathrm{Im}(p) + \mathrm{Re}(p) - j\,\mathrm{Im}(p) = 2\,\mathrm{Re}(p)$$

$$a_1 = (p + p^*) = -2\,\mathrm{Re}(p)$$

$$pp^* = [\mathrm{Re}(p) + j\,\mathrm{Im}(p)][\mathrm{Re}(p) - j\,\mathrm{Im}(p)]$$

$$= \mathrm{Re}^2(p) + \mathrm{Im}^2(p) = |p|^2 \quad \rightarrow \quad a_2 = pp^* = |p|^2$$

$$Ap^* + A^*p = [\mathrm{Re}(A) + j\,\mathrm{Im}(A)][\mathrm{Re}(p) - j\,\mathrm{Im}(p)]$$

$$+ [\mathrm{Re}(A) - j\,\mathrm{Im}(A)][\mathrm{Re}(p) + j\,\mathrm{Im}(p)]$$

$$= 2\,\mathrm{Re}(A)\,\mathrm{Re}(p) + 2\,\mathrm{Im}(A)\,\mathrm{Im}(p)$$

$$Ap^* = [\mathrm{Re}(A) + j\,\mathrm{Im}(A)][\mathrm{Re}(p) - j\,\mathrm{Im}(p)]$$

$$= [\mathrm{Re}(A)\,\mathrm{Re}(p) + \mathrm{Im}(A)\,\mathrm{Im}(p)] + j[\mathrm{Re}(p)\,\mathrm{Im}(A) - \mathrm{Re}(A)\,\mathrm{Im}(p]$$

$$b_1 = -(Ap^* + A^*p) = -2\,\mathrm{Re}(Ap^*)$$

Contoh Soal 19

Tentukan transformasi-Z balik dari : $X(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + 0{,}5z^{-2}}$

Jawab:

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0{,}5z^{-2}} = \frac{b_o + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$b_o = 2\,\mathrm{Re}(A) = 1 \quad \rightarrow \quad \mathrm{Re}(A) = 0{,}5$$

$$a_1 = -2\,\mathrm{Re}(p) = -1 \quad \rightarrow \quad \mathrm{Re}(p) = 0{,}5$$

$$b_1 = 2\,\mathrm{Re}(Ap^*) = 1 \quad \rightarrow \quad \mathrm{Re}(Ap^*) = 0{,}5$$

$$a_2 = |p|^2 = 0{,}5 \quad \rightarrow \quad \mathrm{Re}^2(p) + \mathrm{Im}^2(p) = 0{,}5$$

$$\mathrm{Re}(p) = 0{,}5 \qquad \mathrm{Re}(A) = 0{,}5$$

$$\mathrm{Re}^2(p) + \mathrm{Im}^2(p) = 0{,}25 + \mathrm{Im}^2(p) = 0{,}5$$

$$\mathrm{Im}^2(p) = 0{,}25 \ \rightarrow \ \mathrm{Im}(p) = 0{,}5 \ \rightarrow \ p = 0{,}5 + j0{,}5$$

$$Ap^* = [0{,}5 + j\,\mathrm{Im}(A)](0{,}5 - j0{,}5)$$

$$\mathrm{Re}(Ap^*) = 0{,}25 + 0{,}5\,\mathrm{Im}(A) = 0{,}5$$

$$\mathrm{Im}(A) = 0{,}25 \quad \rightarrow \quad A = 0{,}5 + j0{,}25$$

$$X(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^*z^{-1}}$$

$$= \frac{0{,}5 + j0{,}25}{1 - (0{,}5 + j0{,}5)z^{-1}} + \frac{0{,}5 - j0{,}25}{1 - (0{,}5 - j0{,}5)z^{-1}}$$

$$X(z) = \frac{0,5 + j0,25}{1-(0,5+j0,5)z^{-1}} + \frac{0,5 - j0,25}{1-(0,5-j0,5)z^{-1}}$$

$$0,5 + j0,5 = 0,707e^{j45} \qquad 0,5 - j0,5 = 0,707e^{-j45}$$

$$x(n) = (0,5 + j0,25)(0,707e^{j45})^n + (0,5 - j0,25)(0,707e^{j45})^n$$

$$= (0,5)(0,707)^n(\cos 45n + j\sin 45n)$$

$$+ j(0,25)(0,707^n)(\cos 45n + j\sin 45n)$$

$$+ (0,5)(0,707)^n(\cos 45n - j\sin 45n)$$

$$- j(0,25)(0,707^n)(\cos 45n - j\sin 45n)$$

$$= (0,707)^n \cos 45n - 0,5(0,707)^n \sin 45n$$

TRANSFORMASI-Z SATU SISI

- **Definisi :** $$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Contoh Soal 20

Tentukan transformasi Z satu sisi dari beberapa sinyal diskrit di bawah ini

$$a).\ x_1(n) = \{1,\ 2,\ 5,\ 7,\ 0,\ 1\}$$

$$b).\ x_2(n) = \{1,\ 2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$$

$$c).\ x_3(n) = \{0,\ 1,\ 2,\ 5,\ 7,\ 0,\ 1\}$$

$$d).\ x_4(n) = \{2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$$

Jawab:

a). $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$$X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

b). $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$X_2^+(z) = 5 + 7z^{-1} + z^{-3}$$

c). $x_3(n) = \{0, 1, 2, 5, 7, 0, 1\}$

$$X_3^+(z) = z^{-1} + 2z^{-2} + 5z^{-3} + 7z^{-4} + z^{-7}$$

d). $x_4(n) = \{2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$X_4^+(z) = 5 + 7z^{-1} + z^{-3}$$

Contoh Soal 21

Tentukan transformasi Z satu sisi dari beberapa sinyal impuls di bawah ini

a). $x_5(n) = \delta(n)$

b). $x_6(n) = \delta(n-k), \; k > 0$

c). $x_7(n) = \delta(n+k), \; k > 0$

Jawab:

$$\text{a). } X_5^+(z) = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$$

$$\text{b). } X_6^+(z) = \sum_{n=0}^{\infty} \delta(n-k) z^{-n} = z^{-k}$$

$$\text{c). } X_7^+(z) = \sum_{n=0}^{\infty} \delta(n+k) z^{-n} = 0$$

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- **Time Delay**

$$x(n-k) \rightarrow z^{-k}[X^{+}(z)+\sum_{n=1}^{k} x(-n)z^{n}]$$

Contoh Soal 22

Tentukan transformasi Z satu sisi dari $x_1(n) = x(n-2)$

dimana $x(n) = a^n u(n)$

Jawab:

$$x(n) = a^n u(n) \quad \rightarrow \quad X^{+}(z) = \frac{1}{1-az^{-1}}$$

$$X_1^{+}(z) = z^{-2}\left[X^{+} + \sum_{n=1}^{2} x(-n)z^{n}\right]$$

$$= z^{-2}\left[X^{+} + x(-1)z + x(-2)z^{2}\right]$$

$$= \frac{z^{-2}}{1-az^{-1}} + a^{-1}z^{-1} + a^{-2}$$

Time advance

$$x(n+k) \rightarrow z^k[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n}]$$

Contoh Soal 23

Tentukan transformasi Z satu sisi dari $x_2(n) = x(n+2)$

dimana $x(n) = a^n u(n)$

Jawab:

$$x(n) = a^n u(n) \quad \rightarrow \quad X^+(z) = \frac{1}{1 - az^{-1}}$$

$$X_2^+ = z^2\left[X^+(z) - \sum_{n=0}^{1} x(n)z^{-n}\right]$$

$$= z^2\left[X^+ - x(0) + x(1)z^{-1}\right]$$

$$= \frac{z^2}{1 - az^{-1}} - z^2 - az$$

Contoh Soal 24

Tentukan output dari suatu sistem LTI (Linear Time Invariant) yang dinyatakan oleh persamaan beda :

$$y(n) + 3y(n-1) + y(n-2) = 4{,}5x(n) + 9{,}5x(n-1)$$

$$y(-1) = -8{,}5 \qquad y(-2) = 7{,}5$$

dengan input x(n) = 0 $\longrightarrow$ $y(n) = y_{zi}(n)$

Jawab:

$$Y^{+}(z) + 3z^{-1}[Y^{+}(z) + y(-1)z]$$

$$+ 2z^{-2}[Y^{+}(z) + y(-1)z + y(-2)z^{2}] = 0$$

$$Y^+(z) + 3z^{-1}[Y^+(z) + y(-1)z]$$

$$+ 2z^{-2}[Y^+(z) + y(-1)z + y(-2)z^2] = 0$$

$$Y^+(z)[1 + 3z^{-1} + 2z^{-2}] = -3y(-1) - 2y(-1)z^{-1} - 2y(-2)$$

$$Y^+(z) = \frac{-3(-8,5) - 2(-8,5)z^{-1} - 2(7,5)}{1 + 3z^{-1} + 2z^{-2}}$$

$$= \frac{17z^{-1} + 10,5}{1 + 3z^{-1} + 2z^{-2}}$$

$$\frac{Y^+(z)}{z} = \frac{10,5z + 17}{z^2 + 3z + 2} = \frac{10,5z + 17}{(z+1)(z+2)}$$

$$\frac{Y^{+}(z)}{z} = \frac{10,5z + 17}{z^{2} + 3z + 2} = \frac{10,5z + 17}{(z+1)(z+2)} = \frac{A_1}{z+1} + \frac{A_2}{z+2}$$

$$A_1 = \frac{(z+1)Y^{+}(z)}{z} = \left.\frac{10,5z + 17}{z+2}\right|_{z=-1} = \frac{6,5}{1} = 6,5$$

$$A_2 = \frac{(z+2)Y^{+}(z)}{z} = \left.\frac{10,5z + 17}{z+1}\right|_{z=-2} = \frac{-4}{-1} = 4$$

$$Y^{+}(z) = \frac{6,5z}{z+1} + \frac{4z}{z+2} = \frac{6,5}{1+z^{-1}} + \frac{4}{1+2z^{-1}}$$

$$y_{zi}(n) = 6,5(-1)^{n} + 4(-2)^{n}$$

Contoh Soal 25

Tentukan output dari suatu sistem LTI yang mendapat input x(n) = u(n) dan dinyatakan oleh persamaan beda :

$$y(n) + 6y(n-1) + 8y(n-2) = 5x(n) - 28x(n-1) + 8x(n-2)$$

$$y(-1) = -4 \qquad y(-2) = 3$$

Jawab:

$$Y^{+}(z) + 6z^{-1}[Y^{+}(z) + y(-1)z]$$

$$+ 8z^{-2}[Y^{+}(z) + y(-1)z + y(-2)z^{2}] = 5X^{+}(z)$$

$$- 28z^{-1}[X^{+}(z) + x(-1)z] + 8z^{-2}[X^{+}(z) + x(-1)z + x(-2)z^{2}]$$

$$Y^{+}(z)[1 + 6z^{-1} + 8z^{-2}] - 24 - 32z^{-1} + 24$$

$$= X^{+}(z)[5 - 28z^{-1} + 8z^{-2}]$$

$$Y^{+}(z)[1+6z^{-1}+8z^{-2}]-24-32z^{-1}+24$$

$$=X^{+}(z)[5-28z^{-1}+8z^{-2}]$$

$$Y^{+}(z)[1+6z^{-1}+8z^{-2}]=32z^{-1}+\frac{5-28z^{-1}+8z^{-2}}{1-z^{-1}}=$$

$$Y^{+}(z)=\frac{32z^{-1}-32z^{-2}+5-28z^{-1}+8z^{-2}}{(1+6z^{-1}+8z^{-2})(1-z^{-1})}$$

$$Y^{+}(z)=\frac{5+4z^{-1}-24z^{-2}}{(1+6z^{-1}+8z^{-2})(1-z^{-1})}$$

$$\frac{Y^{+}(z)}{z}=\frac{5z^{2}+4z-24}{(z^{2}+6z+8)(z-1)}$$

$$\frac{Y^{+}(z)}{z} = \frac{5z^2 + 4z - 24}{(z^2 + 6z + 8)(z - 1)} = \frac{5z^2 + 4z - 24}{(z + 2)(z + 4)(z - 1)}$$

$$\frac{5z^2 - 4z - 24}{(z + 2)(z + 4)(z - 1)} = \frac{A_1}{z - 1} + \frac{A_2}{z + 2} + \frac{A_3}{z + 4}$$

$$A_1 = \left.\frac{5z^2 + 4z - 24}{(z + 2)(z + 4)}\right|_{z=1} = \frac{5 + 4 - 24}{(3)(5)} = \frac{-15}{15} = -1$$

$$A_2 = \left.\frac{5z^2 + 4z - 24}{(z + 4)(z - 1)}\right|_{z=-2} = \frac{20 - 8 - 24}{(2)(-3)} = \frac{-12}{-6} = 2$$

$$A_3 = \left.\frac{5z^2 + 4z - 24}{(z + 2)(z - 1)}\right|_{z=-4} = \frac{80 - 16 - 24}{(-2)(-5)} = \frac{40}{10} = 4$$

$$\frac{Y^+}{z} = \frac{-1}{z-1} + \frac{2}{z+2} + \frac{4}{z+4}$$

$$Y^+ = \frac{-1}{1-z^{-1}} + \frac{2}{1+2z^{-1}} + \frac{4}{1+4z^{-1}}$$

$$y(n) = [-1 + 2(-2)^n + 4(-4)^2]u(n)$$